

CONTINUUM PERCOLATION AT AND ABOVE THE UNIQUENESS TRESHOLD ON HOMOGENEOUS SPACES

JOHAN H. TYKESSON

ABSTRACT. We consider the Poisson Boolean model of continuum percolation on a homogeneous space M . Let λ be the intensity of the underlying Poisson process. Let λ_u be the infimum of the set of intensities that a.s. produce a unique unbounded component. First we show that if $\lambda > \lambda_u$ then there is a.s. a unique unbounded component at λ . Then we let $M = \mathbb{H}^2 \times \mathbb{R}$ and show that at λ_u there is a.s. not a unique unbounded component. These results are continuum analogies of theorems by Häggström, Peres and Schonmann.

1. INTRODUCTION AND RESULTS

In this paper we show continuum analogies to some theorems concerning the uniqueness phase in the theory of independent bond and site percolation on graphs. Before turning to our results, we review these theorems.

Let $G = (V, E)$ be an infinite transitive graph with vertex set V and edge set E . Keep each edge with probability p and delete it otherwise, independently for all edges. We call this independent bond percolation on G at level p , and let \mathbf{P}_p be the corresponding probability measure on the subgraphs of G . A connected component in the random subgraph obtained in percolation is called a cluster. Let

$$p_c(G) := \inf\{p : \mathbf{P}_p - \text{a.s. there is an infinite cluster}\}$$

be the critical probability for percolation.

In what follows we will discuss percolation at different levels, and when we do this, we always use the following coupling. To each $e \in E$ we associate an independent random variable U_e which is uniformly distributed on $[0, 1]$. Then say that e is kept at level p if $U_e < p$ and deleted otherwise. Using this construction, we have that if $p_1 < p_2$ then any edge kept at level p_1 is also kept at level p_2 . Therefore we call this coupling the monotone coupling.

Now suppose that $p_c < p_1 < p_2$ and use the monotone coupling. We say that an infinite cluster at level p_2 is p_1 -stable if it contains an infinite cluster at level p_1 . Häggström and Peres [7] showed the following theorem:

Theorem 1.1. *Suppose G is a transitive unimodular graph and that $p_c(G) < p_1 < p_2 \leq 1$. Then any infinite cluster at level p_2 is a.s. p_1 -stable.*

The proof of 1.1 relies on a technique called the mass transport principle, which is not available in the non-unimodular setting. However, Schonmann [10] was able to avoid the use of the mass transport principle and showed:

Theorem 1.2. *Suppose G is a transitive graph and that $p_c(G) < p_1 < p_2 \leq 1$. Then any infinite cluster at level p_2 is a.s. p_1 -stable.*

Theorem 1.2 has the following immediate consequence. Let

$$p_u(G) := \inf\{p : \mathbf{P}_p - \text{a.s. there is a unique infinite cluster}\}$$

be the uniqueness threshold for percolation.

Corollary 1.3. *Suppose G is a transitive graph and that $p > p_u(G)$. Then $\mathbf{P}_p[\text{there is a unique infinite cluster}] = 1$.*

So Corollary 1.3 settles what happens above p_u . But there is also the question what happens at p_u . It turns out that the answer depends on the graph. The following theorem of Peres [9] is of special interest to us:

Theorem 1.4. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two infinite transitive graphs and suppose G is nonamenable and unimodular. Then at $p_u(G \times H)$ there is a.s. not a unique unbounded component.*

In contrast to this result, Benjamini and Schramm [1] showed that on any planar, transitive unimodular graph with one end, there is a.s. a unique infinite cluster at p_u .

We will now discuss analogues of Theorems 1.1, 1.2 and 1.4 in a continuum percolation setting. A Riemannian manifold M is said to be a (Riemannian) homogeneous space if for each $x, y \in M$ there is an isometry that takes x to y . Throughout this paper we assume that M is an unbounded homogeneous space, with metric d_M and volume measure μ_M . When it is clear which space we are working with we will write $d = d_M$ and $\mu = \mu_M$. We let 0 denote the origin of the space.

For one of the main results below it is possible to give a shorter proof under the additional assumption that M is a symmetric space. A connected Riemannian manifold M is said to be a (Riemannian) symmetric space if for each point $p \in M$ there is an isometry I_p such that $I_p(p) = p$ and I_p reverses geodesics through p . The most important symmetric spaces where it makes sense to study continuum percolation are arguably n -dimensional Euclidean space \mathbb{R}^n and n -dimensional hyperbolic space

\mathbb{H}^n . Also products of symmetric spaces are symmetric spaces, for example $\mathbb{H}^2 \times \mathbb{R}$. Any symmetric space is homogeneous. For an example of a noncompact space which is homogeneous but not symmetric, one may consider certain Damek-Ricci spaces, see [2]. Next we introduce the Poisson Boolean model of continuum percolation.

Let $S(x, r) := \{y \in M : d_M(x, y) \leq r\}$ be the closed ball with radius r centered at x . Let X^λ be a Poisson point process on M with intensity λ . Around every point of X^λ we place a ball of unit radius, and denote by C^λ the region of the space that is covered by some ball, that is $C^\lambda := \cup_{x \in X^\lambda} S(x, 1)$. We remark that all proofs below work if we instead consider the model with some arbitrary fixed radius R . Write \mathbf{P}_λ for the probability measure corresponding to this model, which is called the Poisson Boolean model with intensity λ .

Next we introduce some additional notation. Let $V^\lambda := (C^\lambda)^c$ be the vacant region. Let $C^\lambda(x)$ be the component of C^λ containing x . $C^\lambda(x)$ is defined to be the empty set if x is not covered. Let $X^\lambda(A)$ be the Poisson points in the set A . Furthermore denote by $C^\lambda[A]$ the union of all balls centered within the set A . With N_C and N_V we denote the number of unbounded connected components of C^λ and V^λ respectively. The number of unbounded components for the Poisson Boolean model on a homogeneous space is an a.s. constant which equals 0, 1 or ∞ . The proof of this is very similar to the discrete case, see for example Lemma 2.6 in [5]. As in the discrete case, we introduce two critical intensities. Let

$$\lambda_c(M) := \inf\{\lambda : N_C > 0 \text{ a.s.}\} \text{ and } \lambda_u(M) := \inf\{\lambda : N_C = 1 \text{ a.s.}\}$$

be the critical intensity for percolation and the uniqueness threshold for the Poisson Boolean model.

Remark. Obviously it is only interesting to study what happens at and above λ_u when $\lambda_u < \infty$. For example this is case for $\mathbb{H}^2 \times \mathbb{R}$ and may be proved by adjusting the arguments for the \mathbb{H}^2 case, see [11]. Simple modifications (just embed a different graph in the space) of the arguments in Lemma 4.8 in [11] shows that for λ large enough there are a.s. unbounded components in C^λ but a.s. no unbounded components in V^λ . Since any two unbounded components in C^λ must be separated by some unbounded component in V^λ it follows that for λ large enough there is a.s. a unique unbounded component in C^λ .

We will often work with the model at several different intensities at the same time. Suppose we do this at the intensities $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Then we will always assume that $C^{\lambda_{i+1}}$ is the union of C^{λ_i} and balls centered at the points of a Poisson

process with intensity $\lambda_{i+1} - \lambda_i$. We call this the monotone coupling and is obviously the analogy of the discrete coupling described earlier.

Now suppose $\lambda_1 < \lambda_2$ and use the monotone coupling. We say that an unbounded component in C^{λ_2} is λ_1 -stable if it contains some unbounded component in C^{λ_1} . We now state a continuum version of Theorem 1.1.

Theorem 1.5. *Consider the Poisson Boolean model on the homogeneous space M . Suppose $\lambda_c(M) < \lambda_1 < \lambda_2 < \infty$. Then a.s. any unbounded λ_2 -component is λ_1 -stable.*

From Theorem 1.5, the following corollary is immediate.

Corollary 1.6. *Consider the Poisson Boolean model on the homogeneous space M . Suppose $\lambda_u(M) < \lambda$. Then $\mathbf{P}_\lambda[N_C = 1] = 1$.*

Remark. Corollary 1.6 is known in the cases $M = \mathbb{R}^n$ for any $n \geq 2$ (see [8]) and $M = \mathbb{H}^2$ (see [11]).

We will present two proofs of Theorem 1.5. The first is inspired by the proof of Theorem 1.1 and the second is inspired by the proof of Theorem 1.2. To get a continuum analogy to Theorem 1.4 we consider the Poisson Boolean model on a product space.

Theorem 1.7. *Consider the Poisson-Boolean model on $\mathbb{H}^2 \times \mathbb{R}$. At λ_u there is a.s. not a unique unbounded component.*

Note that if one instead considers the model on \mathbb{H}^2 , then Corollary 5.10 in [11] says that at λ_u there is a.s. a unique unbounded component. We now move on to the proofs.

2. UNIQUENESS MONOTONICITY

In this section we first present a short proof for Theorem 1.5 in the symmetric case, and then a proof which only needs the assumption that the space is homogeneous.

First we present an essential ingredient to the first proof, the mass transport principle which is due to Benjamini and Schramm [1]. We denote the group of isometries on the symmetric space M by $\text{Isom}(M)$.

Definition 2.1. *A measure ν on $M \times M$ is said to be diagonally invariant if for all measurable $A, B \subset M$ and $g \in \text{Isom}(M)$*

$$\nu(gA \times gB) = \nu(A \times B).$$

Theorem 2.2. (MASS TRANSPORT PRINCIPLE ON M) *If ν is a positive diagonally invariant measure on $M \times M$ such that $\nu(A \times M) < \infty$ for some open $A \subset M$, then*

$$\nu(B \times M) = \nu(M \times B)$$

for all measurable $B \subset M$.

Actually the mass transport principle is proved in [1] for the case when $M = \mathbb{H}^2$, but as is remarked there, it holds for any symmetric space.

Proof of Theorem 1.5 in the symmetric case: Suppose $\lambda_c < \lambda_1 < \lambda_2$. We couple C^{λ_1} and C^{λ_2} using the monotone coupling. We are done if we can show that any unbounded component of C^{λ_2} contains an unbounded component of C^{λ_1} . Since any ball in C^{λ_1} is also present in C^{λ_2} , this is equivalent to show that any unbounded component of C^{λ_2} intersects an unbounded component of C^{λ_1} . For any point $x \in M$ let

$$D(x) := \inf\{d(x, y) : y \text{ is in an unbounded component of } C^{\lambda_1}\}$$

and let

$$\tilde{D}(x) := \begin{cases} \inf_{y \in C^{\lambda_2}(x)} D(y), & \text{if } x \in C^{\lambda_2} \\ D(x), & \text{otherwise} \end{cases}$$

Define the random set H to be the set of all points x satisfying the conditions

- $C^{\lambda_2}(x)$ is a λ_1 -unstable unbounded component
- $D(x) \leq \tilde{D}(x) + 1/2$

and write $B(x)$ for the event that $x \in H$. Suppose that C^{λ_2} contains an unbounded component which does not intersect an unbounded component of C^{λ_1} . Then this unbounded component contains regions of positive volume in H , so it suffices to show that $\mathbf{P}[B(x)] = 0$. Let $H(x)$ be the connected component of H containing x . Let $B^\infty(x) := B(x) \cap \{\mu(H(x)) = \infty\}$ and $B^f(x) := B(x) \cap \{\mu(H(x)) < \infty\}$. The events B^f and B^∞ partition B . First we show that $\mathbf{P}[B^f(x)] = 0$ using the mass transport principle.

In any unbounded component of C^{λ_2} not intersecting an unbounded component of C^{λ_1} we put mass of unit density. Then all mass in the unbounded component is transported to the regions in the unbounded component which are in H . Let $\nu(A \times B)$ be the expected mass sent from the set A to the set B . Then ν is easily seen to be a positive diagonally invariant measure on $M \times M$. If $\mathbf{P}[B^f(x)] > 0$ then if A is some connected set of finite positive volume, A will get an infinite amount of incoming mass with positive probability, that is $\nu(M \times A) = \infty$. On the other hand, $\nu(A \times M)$, the amount of mass going out from A , is at most $\mu(A) < \infty$. Thus by the mass transport principle $\mathbf{P}[B^f(x)] = 0$.

Next we show $\mathbf{P}[B^\infty(x)] = 0$ by showing $\mathbf{P}[B^\infty(x) | \tilde{D}(x) = r] = 0$ for any r . Fix r . Suppose $\{\tilde{D}(x) = r\}$ happens. Then for $B^\infty(x)$ to happen, there must be infinitely many balls in $C^{\lambda_2}(x)$ centered at distance between $r + 1$ and $r + 1 + 1/2$ from unbounded components in C^{λ_1} . However, this is not possible, as is seen by

“building” up the process as follows. Condition on C^{λ_1} and then on those balls in C^{λ_2} that are centered at distance at least $r + 1$ from unbounded components in C^{λ_1} . We have then not conditioned on the balls that are not present in C^{λ_1} but in C^{λ_2} , and centered at a distance between 0 and $r + 1$ from unbounded components of C^{λ_1} . These balls are centered at a Poisson process of intensity $\lambda_2 - \lambda_1 > 0$ in this region, and this Poisson process is independent of everything else we have previously conditioned on. Thus if there are infinitely many balls in $C^{\lambda_2}(x)$ centered at distance between $r + 1$ and $r + 3/2$ from unbounded components in C^{λ_1} , then balls centered at the points of the previously mentioned Poisson process will almost surely connect $C^{\lambda_2}(x)$ to some unbounded component in C^{λ_1} . Thus $\mathbf{P}[B^\infty(x) | \tilde{D}(x) = r] = 0$ for any r and consequently $\mathbf{P}[B^\infty(x)] = 0$. \square

For the second proof of Theorem 1.5, we need some preliminary results. First we describe a method to find the component of C^λ containing x . This may be considered to be the continuum version of the algorithm described in for example [10] for finding the cluster of a given vertex in discrete percolation.

At x , we grow a ball with unit speed until it has radius 1, when the growth of the ball stops. Whenever the boundary of this ball hits a Poisson point, a new ball starts to grow with unit speed at this point until it has radius 2. In the same way, every time a new Poisson point (which has not already been found) is hit by the boundary of a growing ball, a ball starts to grow at this point until it has radius 2 and so on. Let $L_t^\lambda(x)$ denote the set which has been passed by the boundary of some ball at time t . If $C^\lambda(x)$ is bounded, then $L_t^\lambda(x)$ stops growing at some random time T . In this case $C^\lambda[L_T^\lambda(x)] = C^\lambda(x)$ and $L_T^\lambda(x)$ is the 1-neighbourhood of $C^\lambda(x)$. (If the first ball does not hit any Poisson point, then $C^\lambda(x)$ is the empty set). If $C^\lambda(x)$ is unbounded, then $L_t^\lambda(x)$ never stops growing. We will refer to this procedure to as ”growing the component containing x ”.

In what follows we will make use of the following lemma, which may be considered intuitively clear. The proof is inspired by the proof of the corresponding lemma for the discrete situation which is Lemma 1.1 of [10].

Lemma 2.3. *Consider the Poisson Boolean model on a homogeneous space M . Let $R > 0$ and let $\lambda > \lambda_c$. Any unbounded component of C^λ contains balls of radius R .*

For the proof we need to introduce some further notation. For a connected set A containing x we let $C^\lambda(x, A)$ be all points in A which can be connected to x by some curve in $C^\lambda \cap A$. Let $E_r(x)$ be the union of all balls centered within $S(x, r + 1)$ that are connected to x via a chain of balls centered within $S(x, r + 1)$. Note that $C^\lambda(x, S(x, r)) \subset E_r(x)$.

Let $\delta_r(x) := \sup_{y \in E_r(x) \setminus S(x,r)} d(y, \partial S(x, r))$ where the supremum is defined to be 0 if $E_r(x) \setminus S(x, r)$ is the empty set. Let $\{A \leftrightarrow B\}$ be the event that there is some continuous curve in C^λ which intersects both the set A and the set B . Let A^o be the interior of the set A .

Proof. Fix a point $x \in M$. Since the case $R \leq 1$ is trivial, we suppose $R > 1$. For any $r > 0$ let $F_r(x) := \{x \leftrightarrow \partial S(x, r)\}$ and let

$$G_r(x) := \{C^\lambda(x, S(x, r)) \text{ does not contain a ball of radius } R\}.$$

Let $D_r(x) := F_r(x) \cap G_r(x)$. Let $D(x)$ be the event that x is an unbounded component that does not contain a ball of radius R . Then $D_r(x) \downarrow D(x)$ so it is enough to show that $\mathbf{P}[D_r(x)] \rightarrow 0$ as $r \rightarrow \infty$. Note that $D_r(x)$ is independent of the Poisson process outside $S(x, r+1)$. Also note that $\delta_r(x) \in [0, 2]$.

If $D_r(x) \cap \{\delta_r(x) < 1/2\}$ occurs, then there is a ball centered in $S(x, r-1/2)^o \setminus S(x, r-1)^o$ which is connected to x by a chain of balls centered in $S(x, r-1/2)^o$. All these balls are also included in the set $E_{r-1/2}(x)$, and one of these balls is centered at a distance at most $1/2$ from $\partial S(x, r-1/2)$. This gives

$$D_r(x) \cap \{\delta_r(x) < 1/2\} \subset D_{r-1/2}(x) \cap \{\delta_{r-1/2}(x) \geq 1/2\}. \quad (2.1)$$

We will now proceed by contradiction. Suppose that $\mathbf{P}[D(x)] > 0$ and that $\lim_{r \rightarrow \infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] = 1$. These assumptions imply that

$$\begin{aligned} & \lim_{r \rightarrow \infty} \mathbf{P}[D_r(x) \cap \{\delta_r(x) < 1/2\}] \\ &= \lim_{r \rightarrow \infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] \mathbf{P}[D_r(x)] = \lim_{r \rightarrow \infty} \mathbf{P}[D_r(x)] = \mathbf{P}[D(x)] > 0. \end{aligned}$$

However, by (2.1) we get that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \mathbf{P}[\delta_{r-1/2}(x) \geq 1/2 | D_{r-1/2}(x)] &\geq \limsup_{r \rightarrow \infty} \mathbf{P}[D_{r-1/2}(x) \cap \{\delta_{r-1/2}(x) \geq 1/2\}] \\ &\geq \lim_{r \rightarrow \infty} \mathbf{P}[D_r(x) \cap \{\delta_r(x) < 1/2\}] > 0, \end{aligned}$$

so that in particular $\mathbf{P}[\delta_r(x) \geq 1/2 | D_r(x)]$ does not go to 0 as $r \rightarrow \infty$ which contradicts the assumption $\lim_{r \rightarrow \infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] = 1$. Thus we conclude that $\mathbf{P}[D(x)] = 0$ or/and $\liminf_{r \rightarrow \infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] < 1$. We now assume $\liminf_{r \rightarrow \infty} \mathbf{P}[\delta_r(x) < 1/2 | D_r(x)] < 1$ and show that this implies $\mathbf{P}[D(x)] = 0$. By the assumption, we may pick a constant $c_1 > 0$ and a sequence of positive numbers $\{a_k\}_{k=1}^\infty$ such that $a_{k+1} - a_k \geq 2R + 1$ and $\mathbf{P}[\delta_{a_k}(x) \geq 1/2 | D_{a_k}(x)] \geq c_1$ for all k . On the event $D_{a_k}(x)$ we may pick a point Y on $\partial S(x, a_k + R + 1)$ such that if $S(Y, R + \max(0, 1 - \delta_{a_k}(x)))$ is completely covered by balls centered within $S(Y, R)$, then $D_{a_{k+1}}(x)^c$ occurs since a ball of radius R has been found in $C(x, S(x, a_{k+1}))$.

(this ball is contained in $C(x, S(x, a_{k+1}))$ since $a_{k+1} - a_k \geq 2R + 1$ and $R > 1$). The configuration of balls within $S(Y, R)$ is independent of the Poisson process within $S(x, a_k + 1)$. Now let Δ_k be a random variable with the same distribution as the conditional distribution of $\delta_k(x)$ given the event $D_k(x)$. By the above observations we get that

$$\mathbf{P}[D_{a_{k+1}}(x)^c | D_{a_k}(x)] \geq \mathbf{P}[S(0, R + \max(0, 1 - \Delta_k)) \subset C^\lambda[S(0, R)]] \geq c_2$$

for some constant $c_2 > 0$ for all k . This implies $\lim_{k \rightarrow \infty} \mathbf{P}[D_{a_k}(x)] = 0$ and consequently $\mathbf{P}[D(x)] = 0$. \square

Proof of Theorem 1.5:

We consider the monotone coupling of the model at intensities $\lambda_1 < \lambda_2$, and we write $C = (C^{\lambda_1}, C^{\lambda_2})$. Let

$$E(x) := \{x \text{ is in an unbounded } C^{\lambda_2} \text{ component which is } \lambda_1\text{-unstable.}\}$$

Let

$$E_1(x) := E(x) \cap \{\tilde{D}(x) \leq 3\} \text{ and } E_2(x) := E(x) \cap \{\tilde{D}(x) > 2\},$$

where \tilde{D} is defined as in the proof of Theorem 1.5.

Finally let E , E_1 and E_2 be the events that $E(x)$, $E_1(x)$ and $E_2(x)$ respectively happen for some x .

We will first show that $\mathbf{P}[E_2(x)] = 0$. Pick a and $R = R(a)$ so that

$$\mathbf{P}[S(x, R) \text{ intersects an unbounded component of } C^{\lambda_1}] \geq 1 - a$$

Let $Z' = (Z'^{\lambda_1}, Z'^{\lambda_2})$ and $Z'' = (Z''^{\lambda_1}, Z''^{\lambda_2})$ be two independent copies of C , and let $X' = (X'^{\lambda_1}, X'^{\lambda_2})$ and $X'' = (X''^{\lambda_1}, X''^{\lambda_2})$ be their underlying Poisson processes. A prime will be used to denote objects relating to Z' and a double prime will be used to denote objects relating to Z'' .

Grow the component of Z'^{λ_2} containing x as described above, but if at time t we find that a ball of radius R is contained in $Z'^{\lambda_2}[L_t^{\lambda_2}(x)]$ we stop the process. Let T denote the random time at which the process stops. Note that $T < \infty$ a.s., since if $Z'^{\lambda_2}(x)$ is unbounded, then $Z'^{\lambda_2}(x)$ contains balls of radius R a.s. by Lemma 2.3. Let F_1 be the event that the process stops when a ball of radius R is found, and note that $Z'^{\lambda_2}(x)$ is a.s. bounded on F_1^c . On F_1 , we may (in some way independent of Z'') pick a point Y such that $S(Y, R)$ is covered by $Z'^{\lambda_2}[L_T^{\lambda_2}(x)]$.

For $i = 1, 2$ let

$$X^{\lambda_i} := (X'^{\lambda_i} \cap L_T^{\lambda_2}(x)) \cup (X''^{\lambda_i} \cap L_T^{\lambda_2}(x)^c)$$

and $Z^{\lambda_i} := \cup_{x \in X^{\lambda_i}} S(x, 1)$. In this way, Z^{λ_i} is a Poisson Boolean model with intensity λ_i for $i = 1, 2$, and any ball present in Z^{λ_1} is also present in Z^{λ_2} .

Now put

$$F_2 := F_1 \cap \{S(Y, R) \text{ intersects an unbounded component of } Z''^{\lambda_1}\}.$$

But on F_2 there is some point in $Z^{\lambda_2}(x)$ which is at distance less than or equal to two from some unbounded Z^{λ_1} component, that is $\{\tilde{D}(x) \leq 2\}$ occurs for Z so that $E_2(x)$ does not occur for Z . Since $E_2(x)$ is up to a set of measure 0 contained in F_1 we have that

$$\mathbf{P}[E_2(x)] \leq \mathbf{P}[F_1 \cap F_2^c].$$

Since Z' and Z'' are independent it follows that

$$\mathbf{P}[F_2|F_1] = \mathbf{P}[S(Y, R) \text{ intersects an unbounded component of } Z''^{\lambda_1}] \geq 1 - a$$

and consequently

$$\mathbf{P}[F_1 \cap F_2^c] \leq \mathbf{P}[F_2^c|F_1] < a.$$

Since we may choose a arbitrary small it follows that $\mathbf{P}[E_2(x)] = 0$ as desired.

Next we argue that $\mathbf{P}[E_2(x)] = 0$ for all x implies $\mathbf{P}[E_2] = 0$. Let D be a countable dense subset of M . Then $\mathbf{P}[\cup_{x \in D} E_2(x)] = 0$. But if E_2 occurs then $E_2(x)$ occurs for all x in some unbounded component of C^{λ_2} , in particular for some x in D , so it follows that $\mathbf{P}[E_2(x)] = 0$ implies $\mathbf{P}[E_2] = 0$.

Next we show that $\mathbf{P}[E_1(x)] = 0$. Let $E_1^f(x)$ be the event that $E_1(x)$ occurs and all points in the λ_1 -unstable unbounded C^{λ_2} -component of x which are at distance less than or equal to three from some unbounded C^{λ_1} -component are contained in the ball $S(0, N)$ for some random finite N . Let $E_1^\infty(x)$ be the event that $E_1(x)$ occurs but that there is no such finite N . Let E_1^f and E_1^∞ be the events that $E_1^f(x)$ and $E_1^\infty(x)$ respectively happen for some x .

First we show that $\mathbf{P}[E_1^f] = 0$. Let $E_1^{f,M} := E_1^f \cap \{N \leq M\}$. We will show that $\mathbf{P}[E_1^f] > 0$ implies that $\mathbf{P}[E_2] > 0$. So suppose $\mathbf{P}[E_1^f] > 0$. Then we may pick M so large that $\mathbf{P}[E_1^{f,M}] > 0$. Again let Z' and Z'' be independent with the same distribution as C . Then for $i = 1, 2$ let Z^{λ_i} be the union of all balls from Z'^{λ_i} centered within $S(0, M+1)$ together with the union of all balls from Z''^{λ_i} centered within $S(0, M+1)^c$. Then if $\{Z'^{\lambda_2}[S(0, M+1)] = \emptyset\}$ occurs and $E_1^{f,M}$ occurs for Z'' then E_2 occurs for Z . So since Z' and Z'' are independent we get

$$\mathbf{P}[E_2] \geq \mathbf{P}[Z'^{\lambda_2}[S(0, M+1)] = \emptyset] \mathbf{P}[E_1^{f,M}] > 0$$

which is a contradiction, so $\mathbf{P}[E_1^f] = 0$.

Finally we show that $\mathbf{P}[E_1^\infty] = 0$. However the event $E_1^\infty(x)$ is very similar to the event $B^\infty(x)$ in the first proof of Theorem 1.5, and is shown to have probability 0 in the same way. In the same way it then follows that $\mathbf{P}[E_1^\infty] = 0$. \square

3. CONNECTIVITY

In this section we show how λ_u can be characterized by the connectivity between big balls. This result will be used when we study the model at λ_u on a product space in the next section. Let

$$\lambda_{BB} := \inf\{\lambda : \lim_{R \rightarrow \infty} \inf_{x,y} \mathbf{P}[S(x, R) \leftrightarrow S(y, R) \text{ in } C^\lambda] = 1\}.$$

Note that obviously $\lambda_{BB} \geq \lambda_c$. We will show the following:

Theorem 3.1. *For the Poisson Boolean model on a homogeneous space with $\lambda_u < \infty$ we have $\lambda_u = \lambda_{BB}$.*

The discrete counterpart of this result is Theorem 3.2 of [10], and the proof is similar. The proof is also similar to the second proof of Theorem 1.5 above. First we show that $\lambda_u \leq \lambda_{BB}$.

Proof. Suppose that $\lambda_{BB} < \lambda_1 < \lambda_2$. We will show that at λ_2 there is a.s. a unique unbounded component. For $i = 1, 2$ let

$$A_i(x, y) := \{\mu(C^{\lambda_i}(x)) = \infty, \mu(C^{\lambda_i}(y)) = \infty, C^{\lambda_i}(x) \neq C^{\lambda_i}(y)\},$$

and let

$$A_i := \bigcup_{x,y} A_i(x, y).$$

Since $\lambda_{BB} \geq \lambda_c$ we have by Theorem 1.5 that any unbounded λ_2 component a.s. intersects some unbounded λ_1 component. Therefore

$$\bigcup_{x,y} \{\mu(C^{\lambda_2}(x)) = \infty, \mu(C^{\lambda_2}(y)) = \infty, C^{\lambda_2}(x) \neq C^{\lambda_2}(y)\} \subset A_2 \cup N \quad (3.1)$$

where N is a set of measure 0. In the same way as in the second proof of Theorem 1.5 we have that $\mathbf{P}[A_i(x, y)] = 0$ for all x and y implies $\mathbf{P}[A_i] = 0$. By 3.1, $\mathbf{P}[A_2] = 0$ implies $\mathbf{P}[\text{there is a unique unbounded component at level } \lambda_2] = 1$. Hence it is enough to show that $\mathbf{P}[A_2(x, y)] = 0$ for all x and y .

Definition 3.2. *Suppose C_1 and C_2 are two distinct components in the Poisson Boolean model. A pair of Poisson points $x_1 \in C_1$ and $x_2 \in C_2$ is called a boundary-connection between C_1 and C_2 if $d(x_1, x_2) < 4$ (so that the distance between their corresponding balls is < 2) or there is a sequence of Poisson-points y_1, \dots, y_n such that*

- *the ball centered around y_i intersects the ball centered around y_{i+1} for all i .*
- *y_i is outside C_1 and C_2 for all i .*
- *$d(x_1, y_1) < 4$ and $d(x_2, y_n) < 4$.*

Note that if there is a boundary connection between two components, then at most two more balls are needed to merge them into one component.

If $x, y \in C^{\lambda_1}$ and $C^{\lambda_1}(x) \neq C^{\lambda_1}(y)$, let $B(x, y)$ be the number of boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$. Let

$$\begin{aligned} A_1^0(x, y) &:= A_1(x, y) \cap \{B(x, y) = 0\}, \\ A_1^f(x, y) &:= A_1(x, y) \cap \{B(x, y) < \infty\}, \\ A_1^\infty(x, y) &:= A_1(x, y) \cap \{B(x, y) = \infty\}, \end{aligned}$$

and for $t \in \{0, f, \infty\}$ let A_1^t be the event that $A_1^t(x, y)$ happens for some x and y . In the same way as before it is seen that $\mathbf{P}[A_1^t(x, y)] = 0$ for all x and y implies $\mathbf{P}[A_1^t] = 0$.

Next we will argue that

$$\mathbf{P}[A_1^0(x, y)] = 0 \text{ for all } x \text{ and } y. \quad (3.2)$$

Let Z'^{λ_1} and Z''^{λ_1} be two independent copies of the Poisson Boolean model with intensity λ_1 and let X'^{λ_1} and X''^{λ_1} be their underlying Poisson processes. Since $\lambda_1 > \lambda_{BB}$ we may for any $a > 0$ pick $R = R(a)$ such that

$$\inf_{z_1, z_2} \mathbf{P}_{\lambda_1}[S(z_1, R) \leftrightarrow S(z_2, R)] > 1 - a.$$

Fix x and y and grow the component of x in Z'^{λ_1} (as described earlier) but stop if a ball of radius R is found. Do the same for y . Let F_1 be the event that the processes are stopped when balls of radius R are found, and note that $A_1^0(x, y)$ is up to a set of measure 0 included in F_1 . Let T_x and T_y denote the random times at which the processes are stopped. On F_1 we pick X and Y in some way independent of Z''^{λ_1} such that $S(X, R) \subset Z'^{\lambda_1}[L_{T_x}'^{\lambda_1}(x)]$ and $S(Y, R) \subset Z'^{\lambda_1}[L_{T_y}'^{\lambda_1}(y)]$. Let

$$X^{\lambda_1} := (X'^{\lambda_1} \cap (L_{T_x}'^{\lambda_1}(x) \cup L_{T_y}'^{\lambda_1}(y))) \cup (X''^{\lambda_1} \cap (L_{T_x}'^{\lambda_1}(x) \cup L_{T_y}'^{\lambda_1}(y))^c)$$

and $Z^{\lambda_1} := \cup_{x \in X^{\lambda_1}} S(x, 1)$. The distribution of Z^{λ_1} is by construction the distribution of the Poisson Boolean model with intensity λ_1 . Put

$$F_2 := F_1 \cap \{S(X, R) \leftrightarrow S(Y, R) \text{ in } Z''^{\lambda_1}\}.$$

If we are on F_2 then either $\{Z^{\lambda_1}(x) = Z^{\lambda_1}(y)\}$ occurs or $\{B(x, y) \geq 1\}$ occurs and in neither case we are on $A_1^0(x, y)$. Since

$$\mathbf{P}[F_2|F_1] = \mathbf{P}[S(X, R) \leftrightarrow S(Y, R) \text{ in } Z''^{\lambda_1}] > 1 - a$$

it therefore follows that

$$\mathbf{P}[A_1^0(x, y)] \leq \mathbf{P}[F_1 \cap F_2^c] \leq \mathbf{P}[F_2^c|F_1] < a$$

proving (3.2).

Next we show that

$$\mathbf{P}[A_1^f] = 0. \quad (3.3)$$

Let $A_1^{f,N}$ be the event there are two distinct unbounded components in C^{λ_1} such there are a finite number of boundary connections between them and they are all contained in the ball $S(0, N)$ for some random finite N . Suppose $\mathbf{P}[A_1^f] > 0$ and pick N so large that $\mathbf{P}[A_1^{f,N}] > 0$. Let Z^{λ_1} be the union of the balls from Z'^{λ_1} centered outside $S(0, N)$ and the balls from Z''^{λ_1} centered inside $S(0, N)$. Now suppose that $A_1^{f,N}$ happens for Z'^{λ_1} and that $\{Z''^{\lambda_1}[S(0, N)] = \emptyset\}$ happens. Then we can find two points \tilde{x} and \tilde{y} in separate unbounded components of Z^{λ_1} such that there are no boundary connections between them. It follows by the independence of Z' and Z'' that

$$\mathbf{P}[A_1^0] \geq \mathbf{P}[A_1^{f,N}] \mathbf{P}[Z''^{\lambda_1}[S(0, N)] = \emptyset] > 0,$$

a contradiction which proves (3.3).

Now if $A_1^\infty(x, y)$ happens, then there are infinitely many boundary connections between $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ and a.s. no bounded region contains all boundary connections. Therefore $C^{\lambda_1}(x)$ and $C^{\lambda_1}(y)$ will almost surely have been merged into one unbounded component at level λ_2 by balls that appear in the coupling between level λ_1 and λ_2 . So $\mathbf{P}[A_2(x, y)|A_1^\infty(x, y)] = 0$. Thus, since $A_2(x, y) \subset A_1(x, y)$ and $A_1(x, y)$ is partitioned by $A_1^f(x, y)$ and $A_1^\infty(x, y)$ we conclude

$$\mathbf{P}[A_2(x, y)] = \mathbf{P}[A_2(x, y)|A_1^f(x, y)]\mathbf{P}[A_1^f(x, y)] + \mathbf{P}[A_2(x, y)|A_1^\infty(x, y)]\mathbf{P}[A_1^\infty(x, y)] = 0,$$

for all x and y and so $\lambda_u \leq \lambda_{BB}$.

Next we show the easier result that $\lambda_u \geq \lambda_{BB}$. Suppose $\lambda > \lambda_u$. By Theorem 1.5 there is a.s. a unique unbounded component in C^λ which we denote by C_∞^λ . By the continuum version of the FKG inequality (see [8]) and the fact that there is an isometry mapping x to y it follows that

$$\begin{aligned} \mathbf{P}_\lambda[S(x, R) \leftrightarrow S(y, R)] &\geq \mathbf{P}_\lambda[S(x, R) \text{ and } S(y, R) \text{ intersects } C_\infty^\lambda] \\ &\geq \mathbf{P}_\lambda[S(x, R) \text{ intersects } C_\infty^\lambda]^2. \end{aligned}$$

Since $\lim_{R \rightarrow \infty} \mathbf{P}_\lambda[S(x, R) \text{ intersects } C_\infty^\lambda] = 1$ it follows that $\lambda > \lambda_{BB}$ and thus $\lambda_u \geq \lambda_{BB}$. \square

4. THE SITUATION AT λ_u ON $\mathbb{H}^2 \times \mathbb{R}$

This section is devoted to the proof of Theorem 1.7. We introduce some new notation: if the points $x, y \in \mathbb{H}^2 \times \mathbb{R}$ are in the same component of C^λ then $d_{X^\lambda}(x, y)$ is the smallest number of balls in that component forming a sequence that connects x to y . For a set A we let $C^\lambda(A)$ be the union of all components of C^λ that intersect

A. The length of a curve $\gamma \subset \mathbb{H}^2$ will be denoted by $L(\gamma)$. In this proof $\mu = \mu_{\mathbb{H}^2}$ and $d = d_{\mathbb{H}^2 \times \mathbb{R}}$.

Proof. As noted earlier, it is the case that $\lambda_u(\mathbb{H}^2 \times \mathbb{R}) < \infty$. Suppose that λ_* is such that there is a.s. a unique unbounded component in the Poisson Boolean model with intensity λ_* on $\mathbb{H}^2 \times \mathbb{R}$. We consider the monotone coupling of the model for all intensities below λ_* . We will show that there is some intensity below λ_* that also a.s. produces a unique unbounded component. Denote the unbounded component at λ_* with $C_\infty^{\lambda_*}$. For any $r > 0$, any positive integer n , and any $\lambda \in (0, \lambda_*)$ we define the following three random sets:

$$A_1(r) := \{z \in \mathbb{H}^2 \times \mathbb{R} : S(z, r) \cap C_\infty^{\lambda_*} \neq \emptyset\}$$

$$A_2(r, n) := \{z \in \mathbb{H}^2 \times \mathbb{R} : \sup\{d_{X^{\lambda_*}}(s, t) : s, t \in S(z, r + 1/2) \cap C_\infty^{\lambda_*}\} < n\}$$

$$A_3(r, n, \lambda) := \{z \in \mathbb{H}^2 \times \mathbb{R} : S(z, r + 2n) \cap (X^{\lambda_*} \setminus X^\lambda) = \emptyset\}.$$

Then put

$$A(r, n, \lambda) := A_1(r) \cap A_2(r, n) \cap A_3(r, n, \lambda).$$

Pick $y_1, y_2 \in \mathbb{R}$ and let

$$D := D(y_1, y_2, r, n, \lambda) = \{x \in \mathbb{H}^2 : (x, y_1) \in A(r, n, \lambda) \text{ and } (x, y_2) \in A(r, n, \lambda)\}.$$

Then D is a random set in \mathbb{H}^2 such that the law of D is $\text{Isom}(\mathbb{H}^2)$ -invariant. Next we will show that we can choose the parameters r, n and λ in such a way that D contains unbounded components with positive probability.

To do this, we let \tilde{C} be a Poisson Boolean model in \mathbb{H}^2 with intensity $\tilde{\lambda}$. Let B be a bounded connected set in \mathbb{H}^2 . Choose $\tilde{\lambda}$ so big that \mathbb{H}^2 , we have $\mathbf{E}[L(B \cap \partial\tilde{C})] < \mathbf{E}[\mu(B \cap \tilde{C})]$. By Lemma 5.2 in [11], \tilde{C} contains unbounded components with probability 1. Let $\tilde{C}^D = \tilde{C}^D(y_1, y_2, r, n, \lambda)$ be the union of all balls in \tilde{C} that are completely covered by D .

Suppose E is some bounded connected set in $\mathbb{H}^2 \times \mathbb{R}$. It is clear that

$$\lim_{r \rightarrow \infty} \mathbf{P}[E \subset A_1(r)] = 1, \quad (4.1)$$

and that for fixed r ,

$$\lim_{n \rightarrow \infty} \mathbf{P}[E \subset A_2(r, n)] = 1, \quad (4.2)$$

and that for fixed r and n ,

$$\lim_{\lambda \uparrow \lambda_0} \mathbf{P}[E \subset A_3(r, n, \lambda)] = 1. \quad (4.3)$$

Put $\delta := \mathbf{E}[\mu(B \cap \tilde{C})] - \mathbf{E}[L(B \cap \tilde{C})]$. By (4.1), (4.2) and (4.3) we get that we can find first r_1 big enough, and then n_1 big enough, and finally λ_1 close enough to λ_* so that $\mathbf{E}[\mu(B \cap \tilde{C})] - \mathbf{E}[\mu(B \cap \tilde{C}^D)] < \delta/2$ and $\mathbf{E}[L(B \cap \partial\tilde{C}^D)] - \mathbf{E}[L(B \cap$

$\partial\tilde{C}] < \delta/2$. With these choices of parameters, $\mathbf{E}[\mu(B \cap \tilde{C}^D)] > \mathbf{E}[L(B \cap \tilde{C}^D)]$, so by Lemma 5.2 in [11], we get that \tilde{C}^D contains unbounded components with positive probability. Since $\tilde{C}^D \subset D$, this implies that D contains unbounded components with positive probability. Since the event that D contains unbounded components is $\text{Isom}(\mathbb{H}^2)$ -invariant and determined by the underlying Poisson processes in the model, D contains unbounded components with probability 1.

So we can find an infinite sequence of points $u_1, u_2, \dots \in \mathbb{H}^2$ such that they are all in the same component of D , $d(u_i, u_{i+1}) < 1/2$ for all i and $d(u_1, u_i) \rightarrow \infty$ as $i \rightarrow \infty$. Since $(u_i, y_1) \in A_1$ there is some ball s_i in $C_\infty^{\lambda_0}$ centered within distance $r_1 + 1$ from (u_i, y_1) . Since $d((u_i, y_1), (u_{i+1}, y_1)) < 1/2$ and $(u_i, y_1) \in A_2$ for all i there is a sequence of at most n balls in $C_\infty^{\lambda_0}$ connecting s_i to s_{i+1} . Since the distance between the center of any ball in this sequence and (u_i, y_1) is at most $r_1 + 2n$ and $(u_i, y_1) \in A_3$, all balls in the sequence is present also at level λ_1 . Thus there is an unbounded component in C^{λ_1} that comes within distance r_1 from (u_i, y_1) for all i . In the same way there is an unbounded component in C^{λ_1} that comes within distance r_1 from (u_i, y_2) for all i .

Now choose λ_2 and λ_3 so that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_*$. For $x \in \mathbb{H}^2$ let $D(x)$ be the component of D containing x . Then we have from the above that

$$\mathbf{P}[S((x, y_1), r_1) \leftrightarrow S((x, y_2), r_1) \text{ in } C^{\lambda_2} | \mu(D(x)) = \infty] = 1. \quad (4.4)$$

This follows from the fact that the two unbounded components at level λ_1 above will almost surely be connected by balls appearing in the coupling between level λ_1 and λ_2 . Fix a small and let r_2 be such that for $x \in \mathbb{H}^2$ the ball $S(x, r_2)$ in \mathbb{H}^2 intersects an unbounded component of D with probability at least $1 - a/2$. Let $R = r_1 + r_2$. If $S(x, r_2)$ intersects an unbounded component of D then by (4.4) it follows that a.s. $S((\tilde{x}, y_1), r_1) \leftrightarrow S((\tilde{x}, y_2), r_1)$ in C^{λ_2} for some point $\tilde{x} \in \mathbb{H}^2$ such that $d_{\mathbb{H}^2}(x, \tilde{x}) \leq r_2$, so $S((x, y_1), R) \leftrightarrow S((x, y_2), R)$ in C^{λ_2} . Thus

$$\mathbf{P}[S((x, y_1), R) \leftrightarrow S((x, y_2), R) \text{ in } C^{\lambda_2}] \geq 1 - a/2. \quad (4.5)$$

Fix two points $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ of $\mathbb{H}^2 \times \mathbb{R}$. For $y \in \mathbb{R}$ let

$$F_y := \{S(z_1, R) \leftrightarrow S((u_1, y), R) \text{ in } C^{\lambda_2}\} \cap \{S(z_2, R) \leftrightarrow S((u_2, y), R) \text{ in } C^{\lambda_2}\}$$

By (4.5) we get $\mathbf{P}[F_y] \geq 1 - a$ for all y . In particular it follows that with probability at least $1 - a$ the set $\{y \in \mathbb{R} : F_y \text{ occurs}\}$ is unbounded. But then the set of points in $C^{\lambda_2}(S(z_1, R))$ that come within distance $2R + d_{\mathbb{H}^2}(u_1, u_2)$ from $C^{\lambda_2}(S(z_2, R))$ is unbounded. But if this occurs then some component in C^{λ_2} intersecting $S(z_1, R)$ will a.s. be connected to some component in C^{λ_2} intersecting $S(z_2, R)$ by balls occurring in the coupling between level λ_2 and λ_3 . That is,

$$\mathbf{P}[S(z_1, R) \leftrightarrow S(z_2, R) \text{ in } C^{\lambda_3}] \geq 1 - a.$$

Since a is arbitrary small it follows by Theorem 3.1 there is a.s. a unique unbounded component in C^{λ_3} . \square

Remark. Of course, there is nothing special about \mathbb{R} in the proof of Theorem 1.7. The proof works without any modifications if \mathbb{R} is replaced by any noncompact homogeneous space M such that $\lambda_u(\mathbb{H}^2 \times M) < \infty$. Also, it is possible to show a version of Lemma 5.2 in [11] for \mathbb{H}^n for any $n \geq 3$. Therefore Theorem 1.7 holds for $\mathbb{H}^n \times M$ for any $n \geq 2$ and any noncompact homogeneous space if $\lambda_u(\mathbb{H}^n \times M) < \infty$.

5. FURTHER PROBLEMS

In this section we list some open problems.

1. For which manifolds is $\lambda_u < \infty$?
2. In [11] it is shown that $\lambda_c(\mathbb{H}^n) < \lambda_u(\mathbb{H}^n)$ for any $n \geq 2$ if the radius of the percolating balls is big enough (for $n = 2$ this is shown for any radius). For which manifolds is $\lambda_c < \lambda_u$?
3. For which manifolds with $\lambda_u < \infty$ is there a.s. a unique unbounded component at λ_u ? For which manifolds is there a.s. not a unique unbounded component at λ_u ?

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(Johan H. Tykesson) DEPARTMENT OF MATHEMATICS, DIVISION OF MATHEMATICAL STATISTICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND GÖTEBORG UNIVERSITY, S-412 96
E-mail address, Johan H. Tykesson: `johant@math.chalmers.se`